Non-uniform continuity of periodic Holm-Staley b-family of equations

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Abstract

We consider a family of non-evolutionary partial differential equations known as Holm - Staley b - family which includes the integrable Camassa-Holm and Degasperis-Procesi equations. We show that the solution map is not uniformly continuous. The proof relies on a construction of smooth periodic travelling waves with small amplitude.

1 Introduction

In [17, 18] D. Holm and M. Staley studied an one-dimensional version of an active fluid transport that is described by the following nonlinear equation

$$m_t + um_x + bu_x m = 0, \tag{1.1}$$

with $m = u - u_{xx}$, $u(x,t)$ representing the fluid velocity, while the constant b is a balance or a bifurcation parameter for the solution behavior. It has been shown in [12] that equation (1.1) is integrable only for $b = 2$ and $b = 3$.

In this paper we study the periodic Cauchy problem for the b - family of equations (1.1), namely

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad u(0) = u_0, \quad t \geq 0,\, x \in \mathbb{S}. \tag{1.2}$$

If $b = 2$, then (1.2) becomes the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

Our aim here is to enlarge the result of Himonas and Misiolek [15] (originally proved for the CH equation) for all real $b \neq 0$.

Equation (1.3) was first derived by Fokas and Fuchssteiner [14] as a bi-Hamiltonian system, and then by Camassa and Holm [4] as a model for shallow water waves. The Cauchy problem for the CH equation in both periodic and non periodic case was studied extensively. It has been shown that the Camassa-Holm equation is locally well-posed in H^s, $s > \frac{3}{2}$ with solutions depending continuously on initial data [5, 9, 10, 21, 23]. The Camassa-Holm equation has global solutions but also solutions which blow-up in finite time (see [5, 6, 7, 8, 9, 10, 27]).

When $b = 3$ in (1.2), we recover Degasperis-Procesi (DP) equation which is a model for nonlinear shallow water dynamics,

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.4}$$

The Cauchy problem for the DP equation in both periodic and non periodic case is studied in [13, 22, 25, 26, 28]. For the equations (1.3) and (1.4) the blow-up occurs as wave breaking, that is, the solutions remains bounded but its slope becomes infinite in finite time.

Sometimes, it is more appropriate to consider other version of well-posedness problem, for example if one strengthens the notion of well-posedness, requiring that the mapping data-solution is uniformly continuous. The ill-posedness of some classical nonlinear dispersive equations (for instance Korteweg-de Vries equation, modified Korteweg-de Vries equation, cubic Schrödinger equation, and Benjamin-Ono equation) in both periodic and non-periodic cases are studied in [1, 2, 3]. The approach in these papers is based on the existence and suitable properties of the traveling wave solutions associated to the equations. In particular, a good behavior of their Fourier transforms is required. In [15] Himonas and Misiolek showed that for $s \geq 2$ the solution map $u_0 \to u$ for the CH equation is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S})$ into $C([0,T], \mathrm{H}^s(\mathbb{S}))$. A key step in the proof of that result is a construction of a sequence of smooth travelling waves. Himonas, Kenig and Misiolek [16] extend the result to the range $\frac{3}{2} < s < 2$. Their proof is based on the approximation of solutions by terms containing high and low frequencies and exploring the conservation of the H^1 norm. Note that H^1 norm is a conservation law for equation (1.2) only for $b = 2$.

Recently Gui, Liu, and Tian [24] considered the equation (1.1) on the real line. They proved that the equation is locally well-posed in the Sobolev space $\mathrm{H}^s(\mathbb{R})$ for $s > \frac{3}{2}$. Moreover, they gave the precise blow-up scenario of strong solution of the equation with certain initial data. In [29] Zhou established blow-up results for this family of equations under various classes of initial data. He also proved that the solutions with compact support initial data do not have compact support. In the periodic case, sufficient conditions on the initial data are obtained in [11] to guarantee the finite time blow-up and global existence. Using the ideas from [15], it is also established there the non-uniform continuity of DP equation.

Our main result in this paper is the following.

Theorem 1.1. *For any $s \geq 3$, the solution map $u_0 \to u$ for the equation (1.2) with $b \neq 0$, is not uniformly continuous from any bounded set in $\mathrm{H}^s(\mathbb{S})$ into $\mathrm{C}([0,t_0], \mathrm{H}^s(\mathbb{S}))$. More precisely, for each $s \geq 3$ there exist constants $c_{1,2} > 0$ and two sequences of smooth solutions u_n, v_n of the equation (1.2) such that for any $t \in [0,1]$*

$$\sup_n ||\, u_n(t)||_{\mathrm{H}^s} + \sup_n ||v_n(t)||_{\mathrm{H}^s} \leq c_1,$$

$$\lim_{n\to\infty} ||\, u_n(0) - v_n(0)||_{\mathrm{H}^s} = 0,$$

$$\liminf_n ||\, u_n(t) - v_n(t)||_{\mathrm{H}^s} \geq c_2 \sin\left(\frac{t}{2}\right).$$

The idea of the proof is borrowed from [15]. Two sequences of exact periodic smooth solutions are constructed taking advantage of a scaling property of the b - family. While their initial states converge in H^s - norms, the solutions remain apart at certain time. We use two different parameters but equivalent to those in [15] in order to define appropriate families of solutions. The careful choice of these two parameters is crucial in deriving the H^s estimates. Due to the transcendent dependence on b, here we do not give the sharp estimates for these parameters and merely say that they are sufficiently small.

The paper is organized as follows. In section 2 the periodic travelling waves of (1.2) are studied. Although the corresponding conservative system describing the travelling waves is somehow transcendent and depends on several parameters, the things are arranged so that we study an equivalent Hamiltonian quadratic system for which the conditions for the existence of periodic solutions are more or less known.

The main difficulty here is to establish estimates for the period. This is done in section 3 by calculating the first two terms in the expansion of the period function for periodic travelling waves with small amplitude.

In section 4 we obtain upper estimates for these solutions in H^s - norm and carry on the proof of Theorem 1.1 for $b \neq 0, \pm 1$.

We summarize the corresponding results for the case $b = \pm 1$ in section 5. This approach is not applicable to the case $b = 0$ due to lack of periodic solutions.

2 Periodic travelling waves

In this section we investigate the periodic travelling wave solutions of the b - family equation

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad b \in \mathbb{R}. \tag{2.1}$$

Note that if $u(x,t)$ is a classical solution of (2.1), then such is the function

$$u_c(x,t) = cu(x,ct), \quad \text{for any constant } c.$$

First, take velocity $c = 1$ and look for a travelling-wave solution of (2.1) of the form $u(x,t) = \varphi(x - t)$. One can integrate twice the respective equation

$$-\varphi' + \varphi''' + (b+1)\varphi\varphi' = b\varphi'\varphi'' + \varphi\varphi''' \tag{2.2}$$

to obtain, if $b \neq 1$,

$$|1-\varphi|^{b-1}\left[\varphi'^2 - \varphi^2 + \frac{2C_1}{b-1}\right] = 2C_2, \tag{2.3}$$

and, if $b = 1$,

$$\varphi'^2 - \varphi^2 + 2C_1 \ln|1-\varphi| = 2C_2, \tag{2.4}$$

where C_1, C_2 are constants of integration. Take the general case $b \neq 1$.

In the (X,Y)-plane with $X = \varphi$, $Y = \varphi'$ consider the autonomous system

$$\begin{aligned} \dot{X} &= H_Y/M = 2Y(1-X), \\ \dot{Y} &= -H_X/M = 2X(1-X) + (b-1)(Y^2 - X^2 + d), \end{aligned} \tag{2.5}$$

having a first integral H and an integrating factor M, as follows:

$$H(X,Y) = |\,1-X|^{b-1}(Y^2 - X^2 + d), \quad M(X) = (1-X)|\,1-X|^{b-3}, \tag{2.6}$$

respectively, where it is taken for short $d = 2C_1(b-1)^{-1}$. As well known, system (2.5) has a periodic solution if and only if it has a center. The coordinates (X,Y) of a center of (2.5) must satisfy

$$(1+b)X^2 - 2X + (1-b)d = 0, \;\; Y = 0; \;\; [1-X][1-(b+1)X] < 0. \tag{2.7}$$

One can easily verify the following statement.

Proposition 2.1. *Let $b \neq 0, \pm 1$. System (2.5) has a center if and only if one of the following conditions holds:*

(i) $\quad |\,b| > 1, \quad \dfrac{1}{1-b^2} < d < 1.$

(ii) $\quad |\,b| < 1, \quad 1 < d < \dfrac{1}{1-b^2}.$

(iii) $\quad b < -1, \quad d \geq 1.$

We observe that $\Delta = 1 + d(b^2 - 1) > 0$ for all cases. See the corresponding phase portraits of the systems with a center on Figure 1. Note that, by (2.7), there are no periodic orbits in (2.5) if $b = 0$. Besides, cases $b = \pm 1$ will be considered separately in Section 5. Therefore, we will assume below that $b \neq 0, \pm 1$.

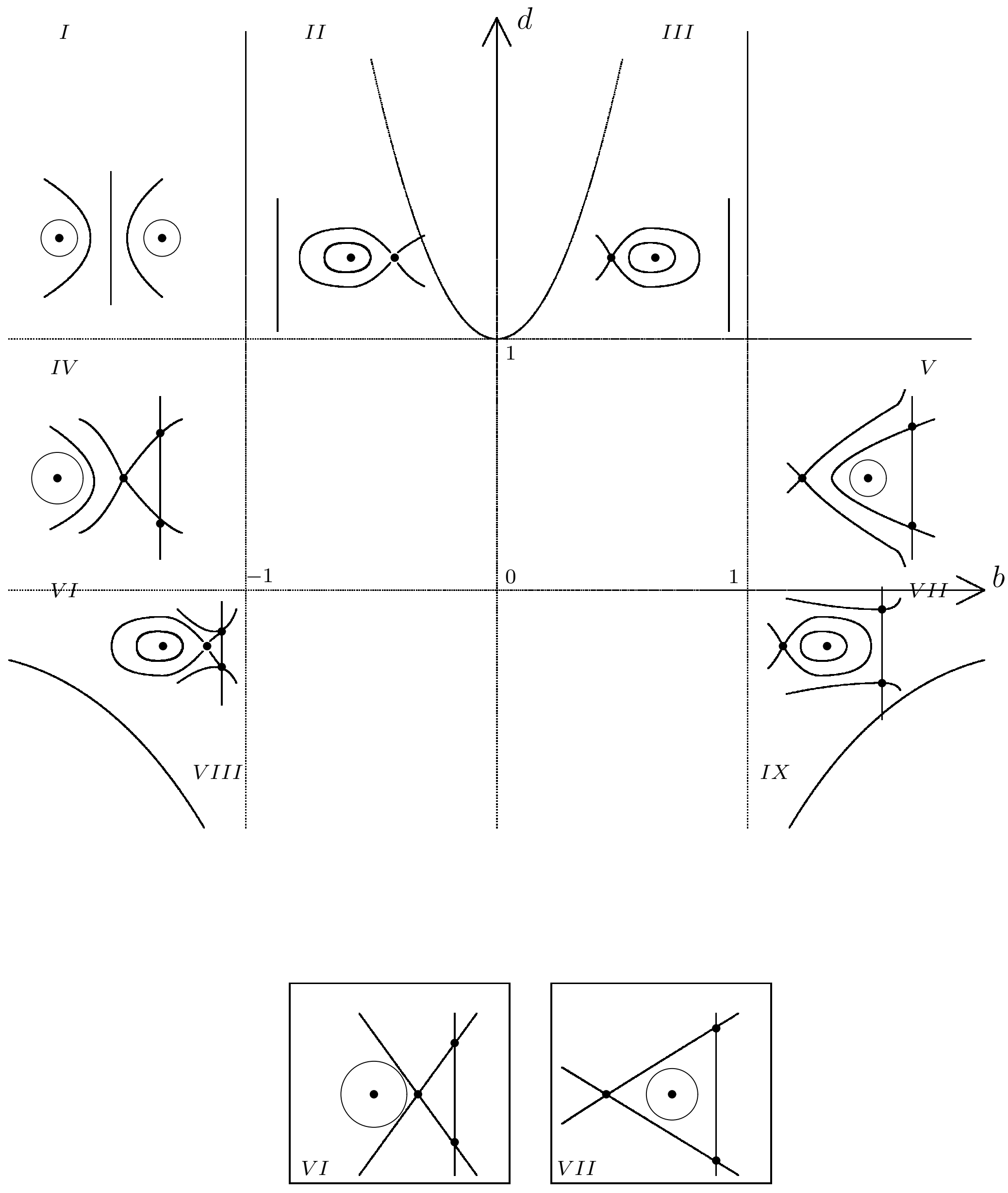

Figure 1: Bifurcation diagram of system (2.5) in the (b, d)-plane. Phase portraits of the systems having periodic solutions are shown only. The vertical invariant line in the (X, Y)-space is always $X = 1$.

The types of quadratic centers are well known (see e.g. [30]). Writing a quadratic system with a center at the origin in the (x, y)-plane as a complex equation with respect to $z = x + iy$, after rescaling we obtain the following types [19]:

$$
\begin{aligned}
&\dot{z} = -iz - z^2 + 2|z|^2 + (A + iB)\bar{z}^2; && \text{Hamiltonian,}\\
&\dot{z} = -iz + Az^2 + 2|z|^2 + B\bar{z}^2; && \text{Reversible,}\\
&\dot{z} = -iz + 4z^2 + 2|z|^2 + (A + iB)\bar{z}^2, \quad A^2 + B^2 = 4; && \text{Codimension 4,}\\
&\dot{z} = -iz + z^2 + (A + iB)\bar{z}^2; && \text{Generalized Lotka-Volterra,}\\
&\dot{z} = -iz + \bar{z}^2; && \text{Hamiltonian triangle.}
\end{aligned}
$$

In the equations above, A and B are real parameters.

By passing to the respective normal form, one can prove the following structure result concerning the types of centers in (2.5).

Proposition 2.2. *Up to an affine transformation of the variables, the center of (2.5) belongs to the following type:*

(i) Hamiltonian triangle, if $b = 2$, $d = 0$;

(ii) Lotka-Volterra, if $|b| > 1$, $b \neq 2$, $d = 0$, *with* $(A, B) = \left(\frac{b}{b-2}, 0\right)$;

(iii) Reversible, if $d \neq 0$, $(b - \sqrt{\Delta})(b + 1) > 0$, *with*

$$
(A, B) = \left(\frac{b\sqrt{\Delta} - 4\sqrt{\Delta} + b}{b(\sqrt{\Delta} - 1)}, \frac{\sqrt{\Delta} + 1}{\sqrt{\Delta} - 1}\right);
$$

(iv) Reversible, if $d \neq 0$, $(b + \sqrt{\Delta})(b + 1) < 0$, *with*

$$
(A, B) = \left(\frac{b\sqrt{\Delta} - 4\sqrt{\Delta} - b}{b(\sqrt{\Delta} + 1)}, \frac{\sqrt{\Delta} - 1}{\sqrt{\Delta} + 1}\right).
$$

We next proceed to determine the interval Σ where the periodic orbits exist. Namely, given b and d as in Proposition 2.1, to find the maximal open interval $\Sigma = \Sigma(b, d)$ such that for any $e \in \Sigma$ the level curve

$$
H(X, Y) = e \tag{2.8}
$$

contains an oval (a simple closed curve without critical points). Clearly, one of the endpoints of Σ is the level e_c corresponding to the center and the other is the level e_s corresponding to the contour at which the period annulus around the center terminates.

Proposition 2.3. *The system (2.5) has a periodic solution for energy levels* $e \in \Sigma = (e_c, e_s)$*, where:*

$$\Sigma = \left(2\frac{|b-\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)-\sqrt{\Delta}-1), 0\right)$$

$$\textit{for} \quad b > 1,\ 0 \le d < 1, \quad \textit{and for} \quad b < -1,\ d \ge 0,$$

$$\Sigma = \left(2\frac{|b-\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)-\sqrt{\Delta}-1), 2\frac{|b+\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)+\sqrt{\Delta}-1)\right)$$

$$\textit{for} \quad |b| > 1,\ \frac{1}{1-b^2} < d < 0 \quad \textit{and for} \quad 0 < b < 1,\ 1 < d < \frac{1}{1-b^2},$$

$$\Sigma = \left(2\frac{|b+\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)+\sqrt{\Delta}-1), 2\frac{|b-\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)-\sqrt{\Delta}-1)\right)$$

$$\textit{for} \quad -1 < b < 0,\ 1 < d < \frac{1}{1-b^2},$$

$$\Sigma = \left(2\frac{|b+\sqrt{\Delta}|^{b-1}}{|b+1|^{b+1}}(d(b+1)+\sqrt{\Delta}-1), 0\right)$$

$$\textit{for} \quad b < -1,\ d > 1.$$

And, finally, if $T = T(e)$, $e \in \Sigma$ is the (minimal) period of the orbit contained in (2.8), one can find the limits $T_c(b,d) = \lim_{e\to e_c} T(e)$ and $T_s(b,d) = \lim_{e\to e_s} T(e)$ (T_s might be infinity). Then, for any T from the open interval with endpoints T_c and T_s, there will be (at least one) periodic orbit of (2.5) having T as a period.

Proposition 2.4. *Let* x_c *be the abscissa of a center of system (2.5). Then*

$$x_c = \frac{1\pm\sqrt{\Delta}}{1+b}, \qquad T_c = 2\pi\sqrt{\frac{1-x_c}{(b+1)x_c-1}}, \qquad T_s = \infty. \tag{2.9}$$

3 The period function for small-amplitude travelling - wave solutions

Below we calculate the first two terms in the expansion of the period function in the case when the periodic wave φ we study has a small amplitude.

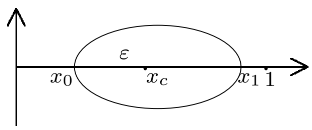

Figure 2. The periodic solution

That is, $x_1 - x_0$ is close to zero where $x_1 = \max\varphi$, $x_0 = \min\varphi$. Therefore the periodic trajectory of (2.5) corresponding to φ is entirely contained in a small neighborhood of a center $(x_c, 0) \in \mathbb{R}^2$, see Proposition 2.4. Let us recall that the period function has an expansion

$$T(\xi) = T_c + T_{2k}\varepsilon^{2k} + T_{2k+1}\varepsilon^{2k+1} + T_{2k+2}\varepsilon^{2k+2} + \ldots$$

with respect to ε, the distance between the center at $(x_c, 0)$ and the intersection point of the orbit with the x-axis $(x_0, 0)$ (see Figure 2, where the case $x_c < 1$ is depicted). The series begins always with an even-degree coefficient T_{2k} for some $k = 1, 2, \ldots$ which is called the k th isochronous constant. Namely, $T_{2k} = 0$ implies also $T_{2k+1} = 0$. If all isochronous constants vanish, all orbits around the center have the same period T_c and the center is isochronous. We will need for our purposes however another "weighted" expansion with respect to $\eta = \varepsilon/(1 - x_c)$ which we are going to handle below.

Proposition 3.1. *The explicit expression of the first isochronous constant is determined from formulas (3.10) and (3.11) below.*

Proof. Take a small positive ε and let $x_0 = x_c - \varepsilon$. Then using (2.6) and (2.8) one obtains by direct calculations

$$\begin{aligned} e = H(x_0, 0) &= |1 - x_c + \varepsilon|^{b-1}[d - (x_c - \varepsilon)^2] \\ &= |1 - x_c|^{b-1}\left(1 + \frac{\varepsilon}{1 - x_c}\right)^{b-1}[d - x_c^2 + 2\varepsilon x_c - \varepsilon^2]. \end{aligned}$$

Then using the identity (equivalent to 2.7)

$$d - x_c^2 = \frac{2x_c(x_c - 1)}{b - 1} \tag{3.1}$$

and denoting $\eta = \varepsilon/(1 - x_c)$, we derive the formula

$$\begin{aligned} e &= (d - x_c^2)|1 - x_c|^{b-1}(1 + \eta)^{b-1}\left[1 - (b - 1)\eta + \frac{(b - 1)(1 - x_c)}{2x_c}\eta^2\right] \\ &= (d - x_c^2)|1 - x_c|^{b-1}e_1(\eta). \end{aligned} \tag{3.2}$$

An expansion in series with respect to η yields immediately that

$$e_1(\eta) = 1 + p_2\eta^2 + p_3\eta^3 + p_4\eta^4 + \ldots, \tag{3.3}$$

$$p_j = \begin{pmatrix} b-1 \\ j \end{pmatrix} - (b-1)\begin{pmatrix} b-1 \\ j-1 \end{pmatrix} + \frac{(b-1)(1-x_c)}{2x_c}\begin{pmatrix} b-1 \\ j-2 \end{pmatrix}, \quad j \geq 2.$$

As x_c is far from zero, we see that all coefficients p_j are uniformly bounded for b fixed for all centers in all regions (I)–(IX), see Fig. 1. Moreover,

$$p_2 = \frac{(b-1)(1-(b+1)x_c)}{2x_c} \neq 0 \text{ and } p_j = \begin{pmatrix} b-1 \\ j-2 \end{pmatrix}\left[p_2 + \tfrac{j-2}{2j}b(b+1)\right], \quad j \geq 3.$$

Similarly, from $e = H(x_0, 0) = H(x_1, 0)$, one can calculate the function ψ in $x_1 = x_c + \psi(\varepsilon)$. Even more conveniently, taking $\psi(\varepsilon) = (1-x_c)\Phi(\eta)$, one obtains as above the following equation for Φ: $e_1(\eta) = e_1(-\Phi)$. Expanding both sides, we derive by calculations the following expansion formula,

$$\Phi(\eta) = \eta + q_2\eta^2 + q_3\eta^3 + q_4\eta^4 + O(\eta^5) \tag{3.4}$$

with

$$q_2 = \frac{p_3}{p_2}, \quad q_3 = \frac{p_3^2}{p_2^2}, \quad q_4 = \frac{2p_3^3 - 2p_2p_3p_4 + p_2^2p_5}{p_2^3}.$$

Let us denote

$$U(x) = e|1-x|^{1-b} + x^2 - d.$$

Then (2.8) becomes $Y^2 = U(X)$ and the period function is determined by

$$T = 2\int_{x_0}^{x_1} \frac{dX}{\sqrt{U(X)}}. \tag{3.5}$$

We perform a change of the variables

$$X = \frac{x_1 - x_0}{2}z + \frac{x_1 + x_0}{2}$$

in (3.5) and obtain

$$T = \int_{-1}^{1} \frac{(x_1 - x_0)dz}{\sqrt{U(x_c + D(z,\eta))}}, \tag{3.6}$$

where

$$D(z,\eta) = (1-x_c)\left(\frac{\Phi(\eta)+\eta}{2}z + \frac{\Phi(\eta)-\eta}{2}\right) := (1-x_c)M(z,\eta).$$

Next, making use of (3.2), we get

$$\begin{aligned} U(x_c + D) &= e|1 - x_c - D|^{1-b} + x_c^2 - d + 2x_cD + D^2 \\ &= (d - x_c^2)\left[e_1(\eta)(1-M)^{1-b} - 1 - (b-1)M - \frac{(b-1)(1-x_c)}{2x_c}M^2\right]. \end{aligned}$$

Conditions $U(x_0) = U(x_1) = 0$ imply that U vanishes for both $M = -\eta$ and $M = \Phi(\eta)$. Hence, using analyticity with respect to M, one can rewrite $U(x_c + D)$ as

$$\begin{aligned} U &= (d - x_c^2)(M + \eta)(\Phi(\eta) - M)(A_0 + A_1 M + A_2 M^2 + \ldots) \\ &= \tfrac{1}{4}(d - x_c^2)(\eta + \Phi(\eta))^2(1 - z^2)(A_0 + A_1 M + A_2 M^2 + \ldots). \end{aligned} \tag{3.7}$$

Comparing the coefficients at the corresponding degrees M^j, $j = 0, 1, 2$, we obtain the following equations for A_j:

$$\begin{aligned} &\eta\Phi A_0 = e_1(\eta) - 1, \\ &\eta\Phi A_1 + (\Phi - \eta)A_0 = (b - 1)(e_1(\eta) - 1), \\ &\eta\Phi A_2 + (\Phi - \eta)A_1 - A_0 = \frac{b(b-1)}{2} e_1(\eta) - \frac{(b-1)(1-x_c)}{2x_c}, \\ &\eta\Phi A_j + (\Phi - \eta)A_{j-1} - A_{j-2} = (-1)^j \begin{pmatrix} 1-b \\ j \end{pmatrix} e_1(\eta), \quad j \geq 3. \end{aligned}$$

By using the expansions (3.3), (3.4) and equality $e_1(\eta) = e_1(-\Phi)$, we calculate

$$\begin{aligned} A_0 &= p_2 + \frac{p_2 p_4 - p_3^2}{p_2}\eta^2 + \frac{p_2 p_3 p_4 - p_3^3}{p_2^2}\eta^3 + O(\eta^4), \\ A_1 &= (b-1)p_2 - p_3 + \frac{(b-1)(p_2 p_4 - p_3^2) + p_3 p_4 - p_2 p_5}{p_2}\eta^2 + O(\eta^3), \\ A_2 &= \frac{b(b-1)}{2}p_2 - (b-1)p_3 + p_4 + O(\eta^2), \\ A_3 &= \frac{b(b^2-1)}{6}p_2 - \frac{b(b-1)}{2}p_3 + (b-1)p_4 - p_5 + O(\eta). \end{aligned}$$

On the other hand, from $M = \frac{1}{2}(\Phi + \eta)z + \frac{1}{2}(\Phi - \eta)$ one obtains

$$\begin{aligned} M &= \eta(1 + \tfrac{1}{2}q_2\eta + \tfrac{1}{2}q_3\eta^2)z + \eta^2(\tfrac{1}{2}q_2 + \tfrac{1}{2}q_3\eta) + O(\eta^4), \\ M^2 &= \eta^2(1 + q_2\eta)z^2 + \eta^3 q_2 z + O(\eta^4), \\ M^3 &= \eta^3 z^3 + O(\eta^4). \end{aligned}$$

Therefore, by direct calculations, we can derive the expression

$$\begin{aligned} &A_0 + A_1 M + A_2 M^2 + A_3 M^3 + O(M^4) \\ &= p_2[1 + a_1 z\eta + (b_0 + b_1 z + b_2 z^2)\eta^2 + (c_0 + c_1 z + c_2 z^2 + c_3 z^3)\eta^3 + O(\eta^4)] \end{aligned}$$

where

$$a_1 = \frac{(b-1)p_2 - p_3}{p_2}, \quad b_0 = \frac{(b-1)p_2 p_3 + 2p_2 p_4 - 3p_3^2}{2p_2^2}, \quad b_1 = \tfrac{1}{2}q_2 a_1,$$

$$b_2 = \frac{b(b-1)p_2 - 2(b-1)p_3 + 2p_4}{2p_2}, \quad c_0 = q_2 b_0, \quad c_2 = q_2 b_2.$$

Next, modulo odd-degree terms with respect to z, one obtains

$$[1 + a_1 z\eta + \ldots]^{-1/2} = 1 + (\tfrac{3}{8}a_1^2 z^2 - \tfrac{1}{2}b_0 - \tfrac{1}{2}b_2 z^2)(\eta^2 + q_2\eta^3) + O(\eta^4). \tag{3.8}$$

Finally, using (3.1) and the definition of Φ we get by direct calculations

$$\frac{1}{4}(d - x_c^2)(\eta + \Phi(\eta))^2 p_2 = \frac{(b+1)x_c - 1}{4(1-x_c)}(x_1 - x_0)^2. \tag{3.9}$$

Therefore, by (3.8), (3.9) and (3.7), one obtains (modulo odd-degree terms)

$$\frac{x_1 - x_0}{\sqrt{U(x_c + D)}} = 2\sqrt{\frac{1-x_c}{(b+1)x_c - 1}} \cdot \frac{1 + (\frac{3}{8}a_1^2 z^2 - \frac{1}{2}b_0 - \frac{1}{2}b_2 z^2)(\eta^2 + q_2\eta^3) + O(\eta^4)}{\sqrt{1-z^2}}$$

and therefore by (3.6)

$$T = 2\pi\sqrt{\frac{1-x_c}{(b+1)x_c - 1}}\left[1 + K(\eta^2 + q_2\eta^3) + O(\eta^4)\right] \tag{3.10}$$

with

$$\begin{aligned} K &= \tfrac{3}{16}a_1^2 - \tfrac{1}{2}b_0 - \tfrac{1}{4}b_2 = \frac{(b^2 - 4b + 3)p_2^2 - 6(b-1)p_2 p_3 - 12 p_2 p_4 + 15 p_3^2}{16 p_2^2} \\ &= \frac{b[2(b-3)(b+1)^2 x_c^2 - 9(b-2)(b+1)x_c + 12(b-1)]}{48[(b+1)x_c - 1]^2}. \end{aligned} \tag{3.11}$$

□

Let us recall that our aim is to obtain sequences of $2\pi/n$-periodic solutions φ_n satisfying appropriate bounds in Sobolev H^s norms. For that purpose, we need the following relations

$$T = \frac{2\pi}{n}, \quad |1 - x_c| = \varepsilon^{2/s} \equiv (x_c - x_0)^{2/s}, \text{where} \quad s \geq 3. \tag{3.12}$$

We first establish the existence of solutions φ of (2.2) satisfying (3.12). Fix $b \neq 0, \pm 1$.

Proposition 3.2. *Given $b \neq 0, \pm 1$ and $s \geq 3$, then there is $N_0 = N_0(b, s)$ sufficiently large, so that for any $n \geq N_0$ there exists a periodic solution $\varphi = \varphi_n$ of (2.2) satisfying (3.12).*

Proof. Clearly, the period T could be small only provided that T_c is small, see (2.9) and (3.10). Therefore, $|1 - x_c| = \varepsilon^{2/s}$ is small and such is $|\eta| = \varepsilon^{1-\frac{2}{s}}$. To calculate K in

(3.11) at first-order approximation, we take $x_c = 1$ to obtain $K = \frac{1}{48}(2b^2 - 11b + 11)$. This implies that $T = T_c = 2\pi/n$, at first-order approximation, which by (2.9) yields

$$x_c = 1 - \frac{b}{n^2} + o(n^{-2}). \tag{3.13}$$

Therefore,

$$\varepsilon = \frac{|b|^{s/2}}{n^s} + o(n^{-s}), \quad \eta = \frac{|b|^{s/2}}{bn^{s-2}} + o(n^{2-s}).$$

Next, by (3.1),

$$d = 1 + \frac{2b^2}{(1-b)n^2} + o(n^{-2}), \quad \Delta = b^2 \left[1 - \frac{2(b+1)}{n^2} + o(n^{-2})\right]. \tag{3.14}$$

We replace this value of d in conditions (i)-(iii) of Proposition 2.1 (neglecting the remainder $o(n^{-2})$) to verify that all they hold, provided that $n^2 > 2(1+b)$. Therefore, Proposition 2.1 holds as long as $n \geq N_0 = N_0(b, s)$ and N_0 is sufficiently large. To verify Proposition 2.3 we need to calculate Σ in any of the cases and check that $e \in \Sigma$. Unfortunately, first-order approximations do not suffice to verify Proposition 2.3. For that reason, we can proceed as follows. Using the above asymptotical values, we conclude that solutions of small period φ_n can exist only for parameters b, d in domains I (right period annulus), II, III and V, see the bifurcation diagram on Figure 1. Therefore, in domains I and V, it suffices to check that $\sqrt{d} < x_0 = x_c - \varepsilon$ because $(\sqrt{d}, 0)$ is the intersection point of the right branch of the invariant hyperbola $y^2 - x^2 + d = 0$ with the abscissa. At first-order approximation, this inequality is equivalent to

$$\sqrt{d} < x_c \quad \Leftrightarrow \quad 1 + \frac{b^2}{(1-b)n^2} < 1 - \frac{b}{n^2}$$

which clearly holds if $|\,b| > 1$. It remains to consider domains II and III where $|\,b| < 1$, $d > 1$. The function $H(x, 0)$ then has just two critical points x_c and x_s (a minimum at x_c and a maximum at x_s) corresponding to the center and the saddle. Moreover,

$$1 < x_c < x_s \quad \text{in II}, \qquad x_s < x_c < 1 \quad \text{in III}.$$

In both cases, $H(x, 0)$ goes to infinity as $x \to 1$ and to minus infinity as $|x| \to \infty$. This information implies that it suffices to prove only

$$e = H(x_0, 0) < H(x_s, 0) = e_s \quad \text{in II, III}, \quad x_0 > 1 \ \text{in II}, \quad x_0 > x_s \ \text{in III}. \tag{3.15}$$

By (2.7), one obtains

$$x_s = \frac{1-b}{1+b} + \frac{b}{n^2} + o(n^{-2}),$$

by (3.2), (3.3) we have

$$e = \frac{2b}{(1-b)n^2} \left|\frac{b}{n^2}\right|^{b-1} [1 + o(1)]$$

and by (2.6)

$$e_s = \frac{4b|\,2b|^{b-1}}{(1+b)^{1+b}}\left[1+O(n^{-2})\right].$$

As $x_0 = x_c$ at first order approximation, all conditions in (3.15) are obviously satisfied. Thus, Proposition 3.2 is proved.

□

So, the solutions $\varphi = \varphi_n(\tau)$ we just constructed have high frequency since $|1-x_c|$ is close to zero. This fact will be used in what follows.

Our next goal is to obtain simple estimates in terms of x_c for the period of the periodic solutions φ having sufficiently small amplitude and high frequency. For $b \neq 0, \pm 1$ an arbitrary but fixed number, such solutions exist in domains I, II, III and V, provided that both d and x_c are close enough to 1 (as shown above). By (3.13), one obtains immediately

$$\frac{|b|}{4n^2} \leq |1-x_c| \leq \frac{4|b|}{n^2}, \quad n \geq N_0(b,s)$$

as long as N_0 is large enough. This is obviously equivalent to

$$\pi\frac{|1-x_c|^{1/2}}{|b|^{1/2}} \leq T \leq 4\pi\frac{|1-x_c|^{1/2}}{|b|^{1/2}}, \quad n \geq N_0(b,s). \tag{3.16}$$

Below, we write $T \simeq |1-x_c|^{1/2}$ for the sake of (3.16).

Finally, let us rewrite equation (2.8) in the form

$$\varphi'^2 = \varphi^2 - d + e(1-\varphi)^{1-b} = U(\varphi), \quad ' = d/d\tau. \tag{3.17}$$

We shall need also the derivatives in the next section

$$\varphi'' = \varphi + \frac{e(b-1)}{2(1-\varphi)^b} = \frac{1}{2}U'(\varphi), \qquad \varphi''' = \left[1+\frac{eb(b-1)}{2(1-\varphi)^{b+1}}\right]\varphi' = \frac{1}{2}U''(\varphi)\varphi'. \tag{3.18}$$

Up to now we have seen that equation (3.17) admits a nonconstant even T - periodic solution (in the corresponding domains of (b,d)) which solves the initial value problem

$$\varphi'' = \varphi + \frac{e(b-1)}{2(1-\varphi)^b}, \quad \varphi(0) = x_0 = x_c - \varepsilon, \quad \varphi'(0) = 0.$$

We conclude this section with an estimate for the incomplete period, proceeding in the same way as above. Take $\alpha \in \left(0, \frac{x_1-x_0}{x_c-x_0}\right)$ and denote

$$\tau(x_0+\alpha\varepsilon) = \int_{x_0}^{x_0+\alpha\varepsilon}\frac{dX}{\sqrt{U(X)}}.$$

Applying the same change of variables, we obtain with

$$\zeta = \frac{2\alpha\varepsilon}{x_1 - x_0} - 1 \in (-1, 1)$$

the formula (instead of (3.6))

$$\tau(x_0 + \alpha\varepsilon) = \frac{1}{2} \int_{-1}^{\zeta} \frac{(x_1 - x_0)dz}{\sqrt{U(x_c + D(z, \eta))}}.$$

Then, including in the calculation of (3.8) all terms up to $O(\eta^2)$, we obtain

$$[1 + a_1 z\eta + \ldots]^{-1/2} = 1 - a_1 z(\tfrac{1}{2}\eta + \tfrac{1}{4}q_2\eta^2) + (\tfrac{3}{8}a_1^2 z^2 - \tfrac{1}{2}b_0 - \tfrac{1}{2}b_2 z^2)\eta^2 + O(\eta^3),$$

instead. Calculating the elementary integral, we get

$$\begin{aligned} \tau(x_0 + \alpha\varepsilon) \;\; &= \sqrt{\frac{1 - x_c}{(b+1)x_c - 1}} \Big\{ (1 + G_1\eta^2) \left(\tfrac{1}{2}\pi + \arcsin\zeta\right) \\ &\quad + \left[a_1(\tfrac{1}{2}\eta + \tfrac{1}{4}q_2\eta^2) - \zeta G_2\eta^2\right] \sqrt{1 - \zeta^2} + O(\eta^3) \Big\}, \end{aligned} \tag{3.19}$$

where $G_1 = \frac{3}{16}a_1^2 - \frac{1}{2}b_0 - \frac{1}{4}b_2, G_2 = \frac{3}{16}a_1^2 - \frac{1}{4}b_2$. Recall that

$$x_1 - x_0 = (1 - x_c)(2\eta + q_2\eta^2 + q_3\eta^3 + \ldots).$$

So, we have

$$\zeta = -1 + \frac{\alpha}{1 + \frac{q_2}{2}\eta + \frac{q_3}{2}\eta^2 + \ldots} = \zeta_0 + \zeta_1\eta + O(\eta^2),$$

where $\zeta_0 = \alpha - 1, \zeta_1 = -\alpha q_2/2$. Substituting this expression into (3.19) gives

$$\tau(x_0 + \alpha\varepsilon) = \frac{|\,1 - x_c|^{1/2}}{\Delta^{1/4}} \left\{ \frac{\pi}{2} + \arcsin\zeta_0 + \eta \left(\frac{\zeta_1}{\sqrt{1 - \zeta_0^2}} + \frac{a_1\sqrt{1 - \zeta_0^2}}{2} \right) + O(\eta^2) \right\}. \tag{3.20}$$

Again, by analyticity argument we can take η or ε small enough that the expression in the brackets in (3.20) can be estimated as follows

$$\frac{|1 - x_c|^{1/2}}{2\Delta^{1/4}} \left(\frac{\pi}{2} + \arcsin(\alpha - 1) \right) \leq \tau(x_0 + \alpha\varepsilon) \leq 2\frac{|1 - x_c|^{1/2}}{\Delta^{1/4}} \left(\frac{\pi}{2} + \arcsin(\alpha - 1) \right). \tag{3.21}$$

We shall use this estimate later in section 4.

4 Non-uniform continuity

In this section we establish appropriate estimates in Sobolev norms of the periodic solutions φ derived in the previous section and then prove our main theorem. The proof of Theorem 1.1 proceeds in the line of [15].

Below, we will use the notation introduced in the previous sections. In the proof of our main theorem, we are going to exploit the properties of small-amplitude high-frequency periodic solutions φ.

First, let us choose the parameter $b \neq 0, \pm 1$ and freeze it. Next, we choose x_c so that $|\,1 - x_c|$ is sufficiently small. And, finally, we choose a periodic orbit sufficiently close to the center $(x_c, 0)$. That is, we choose the parameter e in (2.8) be so close to e_c in order to ensure that the amplitude $x_1 - x_0$ of the corresponding periodic solution φ will satisfy $x_1 - x_0 << 1 - x_c$. Therefore,

$$\varepsilon << 1 - x_c, \quad |\varphi - x_c| \leq |x_1 - x_0|; \text{ and } \frac{|\varphi - x_c|}{|1 - x_c|} << 1. \tag{4.1}$$

In the sequel we need U and several its derivatives evaluated at x_c. Trivial calculations give

$$\begin{aligned} U(x_c) &= P\varepsilon^2 + Q\varepsilon^3 + R\varepsilon^4 + O(\varepsilon^5), \qquad (4.2)\\ U'(x_c) &= \frac{b-1}{1-x_c}U(x_c), \quad U''(x_c) = -2P + \frac{b(b-1)}{(1-x_c)^2}U(x_c), \quad \text{etc.}\dots, \end{aligned}$$

where

$$\begin{aligned} P &= \frac{(b+1)x_c - 1}{1 - x_c}, \quad Q = \frac{(b+1)(2b-3)x_c - 3(b-1)}{3(1-x_c)^2},\\ R &= \frac{(b-2)[(b+1)(b-2)x_c - 2(b-1)]}{4(1-x_c)^3}. \end{aligned}$$

We begin with L^∞ - estimates of the derivatives.

Lemma 4.1. *There exist constants $C_k(b)$, $k \in \mathbb{N}$, so that the following estimates hold*

$$|\varphi^{(k)}| \leq C_k(b) \frac{\varepsilon}{|\,1 - x_c|^{k/2}}. \tag{4.3}$$

Proof. Recall the equation $\varphi'^2 = U(\varphi)$ and its derivatives (3.18). Expanding U around x_c and using the values of U and its derivatives at x_c we calculated earlier in

(4.2), we obtain

$$
\begin{aligned}
|U(\varphi)| &\le |\,U(x_c)| + |\,\varphi - x_c||\,U'(x_c)| + \frac{1}{2}|\,\varphi - x_c|^2|\,U''(x_c)| + O(|\,\varphi - x_c|^3) \\
&\le \left[1 + |\,\varphi - x_c|\frac{|\,b-1|}{|\,1-x_c|} + |\,\varphi - x_c|^2\frac{|\,b||\,b-1|}{2|\,1-x_c|^2}\right]\left[P\varepsilon^2 + |\,Q|\varepsilon^3 + O(\varepsilon^4)\right] \\
&+ \, P|\,\varphi - x_c|^2 + O(|\,\varphi - x_c|^3) \\
&\le P\varepsilon^2\left(5 + |\,b-1| + \frac{|\,b||\,b-1|}{2}\right) + O(\varepsilon^3) \le C(b)P\varepsilon^2
\end{aligned}
$$

because of (4.1). Since $P < |\,b|/|\,1-x_c|$, we obtain the estimate

$$
|\,\varphi'| \le C_1(b)\frac{\varepsilon}{|\,1-x_c|^{1/2}}.
$$

In a similar way, developing $U'(\varphi)$, we verify the estimate

$$
|\,\varphi''| \le C_2(b)\frac{\varepsilon}{|\,1-x_c|}.
$$

Next, we are going to proceed by induction. Taking kth-order derivative of the both sides of (2.2), $k = 0, 1, 2, \ldots$, we obtain the equation

$$
(1-\varphi)\varphi^{(k+3)} = \varphi^{(k+1)} + \sum_{i=0}^{k}\left[c_i\varphi^{(i+1)}\varphi^{(k-i+2)} + d_i\varphi^{(i)}\varphi^{(k-i+1)}\right], \tag{4.4}
$$

where c_i, d_i are certain constants depending on k and b. Applying the induction hypothesis and the first bound from (4.1), we conclude that

$$
|\,1-\varphi||\varphi^{(k+3)}| \le C_k(b)\frac{\varepsilon}{|\,1-x_c|^{(k+1)/2}}.
$$

As $|\,1-\varphi| > |\,1-x_c - O(\varepsilon)| > \frac{1}{2}|\,1-x_c|$, the claim follows.

$\square$

Next we turn to L^2 - estimates.

Lemma 4.2. *There exist constants $D_k(b)$, $k \in \mathbb{N}$, so that the following estimates hold*

$$
||\,\varphi'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} \le D_1(b)\frac{\varepsilon^2}{|\,1-x_c|^{1/2}}. \tag{4.5}
$$

and for any $k = 2, 3, \ldots$

$$
||\,\varphi^{(k)}||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} \le D_k(b)\frac{||\,\varphi'||^2_{L^2[-\frac{T}{2},\frac{T}{2}]}}{|\,1-x_c|^{(k-1)}}.
$$

Proof. For the first derivative, we have

$$|| \varphi' ||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi'^2 d\tau \le C_1^2(b) \frac{\varepsilon^2}{|1-x_c|} T \le D_1(b) \frac{\varepsilon^2}{|1-x_c|^{1/2}}.$$

Next, we get by (3.18), (3.2) and $|1-\varphi| > \frac{1}{2}|1-x_c|$

$$||\varphi''||^2_{L^2[-\frac{T}{2},\frac{T}{2}]} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi''^2 d\tau = -\int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi'''\varphi' d\tau = -\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} U''(\varphi)\varphi'^2 d\tau =$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{b(1-x_c)^b}{(1-\varphi)^{b+1}} e_1(\eta) - 1 \right) \varphi'^2 d\tau \le D_2(b) \frac{||\varphi'||^2_{L^2}}{|1-x_c|}.$$

Finally, we again proceed by induction. Lemma 4.2 holds for $k=2$. By using (4.4), (4.3) and the inductive hypothesis, one easily obtains

$$|| (1-\varphi)\varphi^{(k+3)} || \le || \varphi^{(k+1)} || + \sum_{i=0}^{k} (c_i ||\varphi^{(i+1)}\varphi^{(k-i+2)}|| + d_i || \varphi^{(i)}\varphi^{(k-i+1)}||) \le$$

$$\left[\frac{D_{k+1}}{|1-x_c|^{k/2}} + \sum_{i=0}^{k} \left(\frac{c_i D_{i+1}}{|1-x_c|^{i/2}} . \frac{C_{k-i+2}\varepsilon}{|1-x_c|^{(k-i+2)/2}} + \frac{d_i C_i \varepsilon}{|1-x_c|^{i/2}} . \frac{D_{k-i+1}}{|1-x_c|^{(k-i)/2}} \right) \right] || \varphi' ||$$

$$\le \frac{D_{k+3}}{|1-x_c|^{k/2}} || \varphi' ||.$$

As $|| (1-\varphi)\varphi^{(k+3)} || \ge \frac{1}{2}|1-x_c| || \varphi^{(k+3)} ||$, the statement follows by induction. □

Recall the Sobolev norm

$$|| f ||^2_{\mathrm{H}^s} = \sum_{\xi \in \mathbb{Z}} (1+\xi^2)^s |\hat{f}(\xi)|^2,$$

where $\hat{f}(\xi)$ is the Fourier transform of f.

Lemma 4.3. *Let $\varphi = \varphi_n$ be the $T = \frac{2\pi}{n}$-periodic solution constructed in the end of the previous section. For any $s \ge 3$, there is a positive constant $c_{s,b}$ depending only on s and b, such that*

$$|| \varphi ||^2_{\mathrm{H}^s(-\pi,\pi)} \le c_{s,b} \left(\frac{1}{|1-x_c|^{s-1}} || \varphi' ||^2_{L^2(-\pi,\pi)} + x_1^2 \right).$$

Proof. Let $s = k$, where $k = 3, 4, ...$ Using the facts that

$$|| \varphi^{(k)} ||^2_{L^2(-\pi,\pi)} = n || \varphi^{(k)} ||^2_{L^2(\frac{-\pi}{n},\frac{\pi}{n})} \quad \text{and} \quad x_0 \le \varphi \le x_1$$

these estimates follow from Lemma 4.2.

Let now $s = k + \sigma$, where $k \geq 3$ is a positive integer and $0 < \sigma < 1$. We follow Proposition 3.3 in [15]

$$|| \varphi||^2_{\mathrm{H}^s(-\pi,\pi)} \lesssim || \varphi^{(k)}||^2_{\mathrm{H}^\sigma(-\pi,\pi)} + || \varphi'||^2_{L^2(-\pi,\pi)} + || \varphi||^2_{L^2(-\pi,\pi)}.$$

We have $|| \varphi||^2_{L^2(-\pi,\pi)} = n|| \varphi||^2_{L^2(-\pi/n,\pi/n)} \backsimeq 2\pi x_1^2$. It remains to estimate the H^σ - norm of $\varphi^{(k)}$. It is proven in [15] that for any smooth f the following inequality holds

$$|| f||_{\mathrm{H}^\sigma(-\pi,\pi)} \lesssim || f||^{1-\sigma}_{L^2(-\pi,\pi)}|| f||^\sigma_{\mathrm{H}^1(-\pi,\pi)}.$$

Applying this to $\varphi^{(k)}$ yields

$$|| \varphi^{(k)}||_{\mathrm{H}^\sigma(-\pi,\pi)} \lesssim || \varphi^{(k)}||^{1-\sigma}_{L^2(-\pi,\pi)}|| \varphi^{(k)}||^\sigma_{\mathrm{H}^1(-\pi,\pi)}.$$

Since $|1 - x_c| < 1$, using the estimates from Lemma 4.2 we obtain

$$|| \varphi^{(k)}||_{\mathrm{H}^1(-\pi,\pi)} \backsimeq || \varphi^{(k)}||_{L^2(-\pi,\pi)} + || \varphi^{(k+1)}||_{L^2(-\pi,\pi)} \lesssim$$
$$\left(\frac{1}{|1 - x_c|^{(k-1)/2}} + \frac{1}{|1 - x_c|^{k/2}}\right) || \varphi'||_{L^2(-\pi,\pi)} \lesssim \frac{|| \varphi'||_{L^2(-\pi,\pi)}}{|1 - x_c|^{k/2}}.$$

Combining these inequalities, we get

$$|| \varphi^{(k)}||_{\mathrm{H}^\sigma(-\pi,\pi)} \lesssim \frac{|| \varphi'||^{1-\sigma}_{L^2(-\pi,\pi)}}{|1 - x_c|^{(k-1)(1-\sigma)/2}} \frac{|| \varphi'||^\sigma_{L^2(-\pi,\pi)}}{|1 - x_c|^{k\sigma/2}} \lesssim \frac{|| \varphi'||_{L^2(-\pi,\pi)}}{|1 - x_c|^{(k+\sigma-1)/2}}$$

from where the lemma follows.

□

Proof of Theorem 1.1. Let $s \geq 3$ and let φ_n be the $2\pi/n$ - periodic smooth solution, constructed above. Recall from section 3 that

$$n \backsimeq \frac{1}{|1 - x_c|^{1/2}}. \tag{4.6}$$

Consider the following two sequences of travelling wave solutions

$$u_n(x, t) = \varphi_n(x - t), \qquad v_n(x, t) = c_n\varphi_n(x - c_nt) \tag{4.7}$$

and take

$$c_n = 1 + \frac{1}{n}.$$

As in [15] we show that these sequences are bounded, their difference goes to zero at time $t = 0$ and stays apart from zero at $t > 0$.

The boundedness and the limit at the time $t = 0$ are almost straightforward. Taking into account (4.5), (4.6) it is obtained

$$|| \varphi_n' ||^2_{L^2(-\pi,\pi)} = n|| \varphi_n' ||^2_{L^2(\frac{-\pi}{n},\frac{\pi}{n})} \lesssim \frac{1}{|1-x_c|^{1/2}} \frac{D_1(b)\varepsilon^2}{|1-x_c|^{1/2}}.$$

Also, we have from (4.7) and Lemma 4.3 that

$$||v_n(t)||^2_{\mathrm{H}^s(-\pi,\pi)} = c_n^2|| \varphi_n ||^2_{\mathrm{H}^s(-\pi,\pi)} \lesssim c_n^2 c_{b,s} \frac{\varepsilon^2}{|1-x_c|^s} + x_1^2,$$

where $s \geq 3$. The choice of parameters $|1-x_c|^s = \varepsilon^2$ assures that the both sequences u_n and v_n are bounded in H^s - norms.

Further,

$$|| v_n(0) - u_n(0)||^2_{\mathrm{H}^s(\mathbb{S})} = || c_n\varphi_n - \varphi_n||^2_{\mathrm{H}^s(\mathbb{S})} = (c_n-1)^2|| \varphi_n||^2_{\mathrm{H}^s(\mathbb{S})} \cong \frac{1}{n^2},$$

which goes to 0 when $n \to \infty$.

Finally, the behavior at time $t > 0$ can be established in the following way.

$$|| v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} = \sum_{\xi\in\mathbb{Z}} (1+\xi^2)^s | c_n\widehat{\varphi_n}(\cdot - c_n t)(\xi) - \widehat{\varphi_n}(\cdot - t)(\xi)|^2,$$

where $\widehat{\varphi_n}(\cdot - c_n t)(\xi)$ is the Fourier transform of the function $\varphi_n(x - c_n t)$ with respect to x, that is after changing the variables $\widehat{\varphi_n}(\cdot - c_n t)(\xi) = e^{-itc_n\xi}\widehat{\varphi_n}(\xi)$. Hence,

$$|| v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} = \sum_{\xi\in\mathbb{Z}} (1+\xi^2)^s \left| (e^{\frac{-it\xi}{n}} - 1) + \frac{1}{n} e^{\frac{-it\xi}{n}} \right|^2 | \widehat{\varphi}(\xi)|^2,$$

and

$$|| v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \geq (1+n^2)^s |(e^{-it} - 1) + \frac{1}{n} e^{-it}|^2 | \hat{\varphi}_n(n)|^2.$$

Since φ_n is a $2\pi/n$ - periodic, even function and after integrating by parts, we get

$$\hat{\varphi}_n(n) = \frac{n}{\sqrt{2\pi}} \int_{-\pi/n}^{\pi/n} e^{-in\tau}\varphi_n(\tau)d\tau = -\frac{2}{\sqrt{2\pi}} \int_0^{\pi/n} \sin(n\tau)\varphi'(\tau)d\tau.$$

Therefore

$$|| v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \geq \frac{2}{\pi}(1+n^2)^s |(e^{-it} - 1) + \frac{1}{n} e^{-it}|^2 | B_n|^2, \tag{4.8}$$

where we denote

$$B_n = \int_0^{\pi/n} \sin(n\tau)\varphi_n'(\tau)d\tau.$$

The integral for B_n can be estimated from below in the same line as Lemma 4.1 in [15].

Lemma 4.4. *There exists a constant $c_0 > 0$ independent of n such that*

$$B_n \geq c_0\varepsilon.$$

Proof. We have

$$B_n = \int_0^{\pi/n} \sin(n\tau)\varphi_n'(\tau)d\tau = \int_{x_0}^{x_1} \sin(n\tau(\varphi))d\varphi.$$

For any $\alpha \in (0, \frac{x_1-x_0}{x_c-x_0})$,

$$B_n \geq \int_{x_0+\alpha\varepsilon/2}^{x_0+\alpha\varepsilon} \sin(n\tau(\varphi))d\varphi. \tag{4.9}$$

We take α to satisfy the condition

$$n\tau(x_0+\alpha\varepsilon) \leq \frac{\pi}{2}. \tag{4.10}$$

To do this, let us first recall the estimate (3.16) on the period. Next, by (3.14), we have $\Delta = b^2[1+O(n^{-2})]$, therefore one can rewrite (3.16) as

$$\pi\frac{|1-x_c|^{1/2}}{\Delta^{1/4}} \leq T = \frac{2\pi}{n} \leq 4\pi\frac{|1-x_c|^{1/2}}{\Delta^{1/4}}.$$

Further, taking advantage from the estimate on the incomplete period (3.21), we get

$$\frac{1}{4}\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right) \leq n\tau(x_0+\alpha\varepsilon) \leq 4\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right). \tag{4.11}$$

Thus, to satisfy the condition (4.10) we take α so that

$$4\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right) = \frac{\pi}{2},$$

or $\arcsin(\alpha-1) = -\frac{3\pi}{8}$. With this choice of α inequality (4.9) gives

$$\begin{aligned} B_n &\geq \int_{x_0+\alpha\varepsilon/2}^{x_0+\alpha\varepsilon} \sin(n\tau(x_0+\frac{\alpha}{2}\varepsilon))d\varphi \\ &= \sin\left(n\tau(x_0+\frac{\alpha}{2}\varepsilon)\right)\frac{\alpha}{2}\varepsilon \geq \left[\frac{\alpha}{2}\sin\left(\frac{1}{4}\left(\frac{\pi}{2}+\arcsin(\frac{\alpha}{2}-1)\right)\right)\right]\varepsilon, \end{aligned}$$

where the last inequality follows from the lower bound in (4.11) and α is replaced by $\alpha/2$. This proves the lemma.

□

Returning to (4.8) one gets

$$||\, v_n(t) - u_n(t)||^2_{\mathrm{H}^s(S)} \gtrsim n^{2s}\varepsilon^2 \left|(e^{-it}-1)+\frac{1}{n}e^{-it}\right|^2.$$

Thus, the desired estimate is obtained as in [15] using (4.6) and $|1-x_c|^s = \varepsilon^2$.

■

5 The cases $b = \pm 1$.

Here we study the cases $b = \pm 1$ in the Holm - Staley equation (1.1). Since most of the computations and estimates are similar to those in sections 2, 3 and 4, we give only the key results and differences.

5.1 The case $b = -1$.

Equation (1.2) with $b = -1$ has no hydrodynamical relevance, but we consider it here due to its simplicity. One should start with it, because all things are transparent. By (2.3), we obtain the conic curve

$$(\varphi - 1)^{-2}(\varphi'^2 - \varphi^2 + d) = e. \tag{5.1}$$

There are periodic solutions φ for $d > 1$ and $e \in \left(\frac{d}{1-d}, -1\right)$. They surround the center at $(d, 0)$. One can rewrite (5.1) as

$$\varphi'^2 + \frac{1}{e}(\varphi - 1)^2 - 2(\varphi - 1) + d = 0,$$

with new d and e (equal to $d - 1$ and $-(e+1)^{-1}$, respectively). Hence, periodic solutions exist for $e > d > 0$. Denote $A = \sqrt{e(e-d)}$. Then they are given explicitly by

$$\varphi(\tau) = 1 + e - A\cos\frac{\tau}{\sqrt{e}}$$

with period $T = 2\pi\sqrt{e}$. Assuming A and e small, one can find integer n such that $n \simeq \frac{1}{\sqrt{e}}$ and $T = \frac{2\pi}{n}$.

As before, let us take the following two sequences of solutions

$$u_n(x,t) = \varphi_n(x - t), \quad v_n(x,t) = c_n\varphi_n(x - c_n t), \quad c_n = 1 + \frac{1}{n}.$$

It is sufficient to estimate v_n. A direct computation gives

$$||\, v_n(t)||^2_{\mathrm{H}^s(-\pi,\pi)} = c_n^2\left[(1+e)^2 + \frac{1}{4}(1+n^2)^s A^2\right] \le 4\left[(1+e)^2 + 2^{s-2}n^{2s}A^2\right].$$

Boundedness is achieved upon the condition $A^2 = e^s, s \ge 3$. The limit at $t = 0$ is the same as above. It remains to consider the estimate

$$||\, v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \ge (1+n^2)^s|(e^{-it} - 1) + \frac{1}{n}e^{-it}|^2|\,\hat{\varphi}_n(n)|^2.$$

Trivial computations yield that $\hat{\varphi}_n(n) = -A/2$, so

$$||\, v_n(t) - u_n(t)||^2_{\mathrm{H}^s(\mathbb{S})} \ge \frac{1}{4}A^2 n^{2s}|(e^{-it} - 1) + \frac{1}{n}e^{-it}|^2 = \frac{1}{4}|(e^{-it} - 1) + \frac{1}{n}e^{-it}|^2$$

in view of relation $A^2 = e^s, s \ge 3$. Hence, the result follows as in [15].

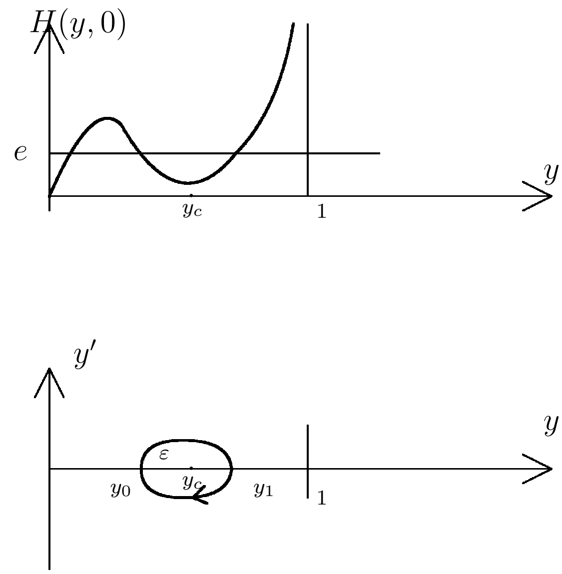

Figure 3: The periodic solution of (5.2)

5.2 Case $\mathbf{b = 1}$.

By (2.4), travelling-wave solutions of (1.1) of the form $u = y(x-t)$, $y < 1$ will satisfy

$$H(y, y') \equiv y'^2 - y^2 - 2d\ln(1-y) = e. \tag{5.2}$$

Hence, we have a conservative system with a Newtonian first integral H. The critical points of the potential $H(y,0)$ are

$$y_c = \frac{1+\sqrt{1-4d}}{2}, \quad y_s = \frac{1-\sqrt{1-4d}}{2}, \quad d < 1/4,$$

where c stands for the center and s for the saddle. It is straightforward to verify that for $d \in (0, 1/4)$ and $H(y_c, 0) < e < H(y_s, 0)$ there are periodic solutions (see also Figure 3).

Let $y_1 = \max y$ and $y_0 = \min y$. We assume that $y_1 - y_0$ is small. So, there is a periodic solution to (5.2) which satisfies the following initial value problem

$$y'' = y - \frac{d}{1-y}, \quad y(0) = y_0, \quad y'(0) = 0.$$

The period function has an expansion

$$T(\varepsilon) = T_c + \varepsilon^2 T_2 + \ldots,$$

where ε is defined as

$$y_0 = y_c - \varepsilon \quad \text{and} \quad T_c = 2\pi\sqrt{\frac{1-y_c}{2y_c-1}}.$$

Note that, when $d < 1/4$, then $y_c > 1/2$. Similar computations as in Section 2 give

$$e = H(y_0, 0) = -y_c^2 - 2d\ln(1-y_c) + P\varepsilon^2 + Q\varepsilon^3 + R\varepsilon^4 + \dots,$$

where

$$P = \frac{2y_c - 1}{1-y_c}, \quad Q = -\frac{2y_c}{3(1-y_c)^2}, \quad R = \frac{y_c}{2(1-y_c)^3}.$$

In terms of η the energy e becomes

$$e = H(y_0, 0) = -y_c^2 - 2d\ln(1-y_c) + p_2\eta^2 + p_3\eta^3 + p_4\eta^4 + \dots,$$

where

$$p_2 = (2y_c - 1)(1-y_c), \quad p_3 = -\frac{2}{3}y_c(1-y_c), \quad p_4 = \frac{y_c}{2}(1-y_c).$$

For these calculations we have used the identity $y_c - d/(1-y_c) = 0$. From $H(y_0, 0) = H(y_1, 0)$ one obtains that

$$y_1 = y_c + (1-y_c)\Phi(\eta), \quad \Phi(\eta) = \eta + r\eta^2 + r^2\eta^3 + \dots$$

with $r = p_3/p_2$. Denote $U(y) = y^2 + 2d\ln(1-y) + e$. Then (5.2) becomes $y'^2 = U(y)$ and the period function is

$$T = 2\int_{y_0}^{y_1} \frac{dy}{\sqrt{U(y)}}.$$

As before we put

$$y = \frac{y_1 - y_0}{2}z + \frac{y_1 + y_0}{2},$$

thus

$$T = \int_{-1}^{1} \frac{(y_1 - y_0)\, dz}{\sqrt{U(y_c + D(z, \eta))}}.$$

In the same line of computations we obtain the formula

$$T = 2\pi\sqrt{\frac{1-y_c}{2y_c-1}}\left(1 + \frac{y_c(9-8y_c)}{24(2y_c-1)^2}\eta^2 + \dots\right). \tag{5.3}$$

As above one can take η so small, that the expression in the brackets in (5.3) will take values in $[\frac{1}{2}, 2]$. This gives

$$\pi\sqrt{\frac{1-y_c}{2y_c-1}} \le T \le 4\pi\sqrt{\frac{1-y_c}{2y_c-1}}.$$

We write $T \simeq \sqrt{1-y_c}$ and for any sufficiently large integer n one can find y_c so that and $1-y_c$ is sufficiently small in order to achieve

$$T = \frac{2\pi}{n} \quad \text{and} \quad n \simeq \frac{1}{\sqrt{1-y_c}}.$$

Hence, we have constructed high-frequency solution $y = y_n(t)$ with period $T = 2\pi/n$.

Next, in order to estimate the incomplete integral $\tau(y_0+\alpha\varepsilon)$, by long but straightforward computations similar to those in section 3 we obtain

$$\frac{1}{2}\sqrt{\frac{1-y_c}{2y_c-1}}\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right) \le \tau(y_0+\alpha\varepsilon) \le 2\sqrt{\frac{1-y_c}{2y_c-1}}\left(\frac{\pi}{2}+\arcsin(\alpha-1)\right).$$

Finally, we need some estimates in order to obtain upper bounds for these solutions. We need writing (5.2) in the form $y'^2 = U(y)$ and then calculate the derivatives

$$y'' = \frac{1}{2}U'(y), \qquad y''' = \frac{1}{2}U''(y)y'. \tag{5.4}$$

Assume that y_c is close enough to 1 and $y_1 - y_0 << 1 - y_c$, and also

$$\varepsilon << 1-y_c, \quad |y-y_c| \le |y_1-y_0| \ \text{ and } \ \frac{|y-y_c|}{1-y_c} << 1. \tag{5.5}$$

Expanding U around y_c, using (5.5) and that $P \le 1/(1-y_c)$ we obtain the estimate

$$|y'| \le \frac{\sqrt{10}\,\varepsilon}{(1-y_c)^{1/2}}.$$

In a similar way, developing $U'(y)$ we get $|y''| \le 4\varepsilon/(1-y_c)$. Again, induction arguments give the estimates

$$|y^{(k)}| \le C_k\frac{\varepsilon}{(1-y_c)^{k/2}}.$$

From the above expressions we obtain L^2-estimate for the first derivative

$$||y'||^2 = \int_{-T/2}^{T/2} y'^2 d\tau \le C_1\frac{\varepsilon^2}{1-y_c}T \le D_1\frac{\varepsilon^2}{(1-y_c)^{1/2}}.$$

Finally, from (5.4) we obtain an estimate for the second derivative

$$||y''||_{L^2}^2 = \int_{-\frac{T}{2}}^{\frac{T}{2}} y''^2 d\tau = -\int_{-\frac{T}{2}}^{\frac{T}{2}} y'''y' d\tau = -\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} U''(y)y'^2 d\tau \le D_2\frac{||y'||_{L^2}^2}{1-y_c}.$$

Now, we can proceed by induction to obtain similar estimates for the higher-order derivatives as in Lemma 4.2. The proof of Theorem 1.1 is then finished in the same way as in the general case.

6 Conclusions

In this paper we study the Cauchy problem for the periodic Holm - Staley b - family of equations. The results by Himonas and Misiolek [15] (proved for the CH equation $b = 2$ only) and the one for the DP equation $b = 3$ [11], are extended for the general case of b-family $b \neq 0$ (Theorem 1.1). We show that the solution map is not uniformly continuous in $\mathrm{H}^s, s \geq 3$. The proof is based on the construction of suitable smooth periodic solutions of small amplitude. To our knowledge, this idea comes from Kato [20].

Our result for the whole b-family is weaker than the results for particular values of b in the above mentioned papers [15] and [11] where $s \geq 2$. This is because we assume that the small parameters ε and $|1-x_c|$ are related as $\varepsilon << |1-x_c|$. We need this assumption in order to estimate the period, which is the main difficulty here. Then the relation $|1-x_c|^s = \varepsilon^2$ is valid for $s > 2$.

From the other hand, an interpolation argument is used to obtain the estimates for non-integer Sobolev indexes. That is the reason why the range $2 < s < 3$ is not covered. Perhaps, one should consider the case $s = 2$ separately, but this makes the estimates of the period for arbitrary b more difficult.

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